Rational Expression Concepts -- Introduction

This module provides an introduction to rational expressions.

The term “rational” in math is not used in the sense of “sane” or “sensible.” It is instead used to imply a **ratio**, or fraction. A **rational expression** is the ratio of two polynomials: for instance, —— is a rational expression.

There are two rules for working with rational expressions.

1. Begin every problem by factoring everything you can.
2. Remember that, despite all the complicated looking functions, a rational expression is just a fraction: you manipulate them using all the rules of fractions that you are familiar with.

Rational Expression Concepts -- Simplifying Rational Expressions
This module provides techniques for simplifying rational expressions.

How do you **simplify** a fraction? The answer is, you divide the top and bottom by the same thing.

Equation:

$$\frac{4}{6}=\frac{4\div 2}{6\div 2}=\frac{2}{3}$$

So $\frac{4}{6}$ and $\frac{2}{3}$ are two different ways of writing the same number.

On the left, a pizza divided into six equal slices: the four shaded-in regions represent $\frac{4}{6}$ of a pizza. On the right, a pizza divided into three equal slices: the two shaded-in regions represent $\frac{2}{3}$ of a pizza. The two areas are identical: $\frac{4}{6}$ and $\frac{2}{3}$ are two different ways of expressing the **same amount of pizza**.

In some cases, you have to repeat this process more than once before the fraction is fully simplified.

Equation:

$$\frac{40}{48}=\frac{40\div 4}{48\div 4}=\frac{10}{12}=\frac{10\div 2}{12\div 2}=\frac{5}{6}$$

It is vital to remember that **we have not divided this fraction by 4, or by 2, or by 8**. We have rewritten the fraction in another form: $\frac{40}{48}$ is the **same number** as $\frac{5}{6}$. In strictly practical terms, if you are given the choice

between $\frac{40}{48}$ of a pizza or $\frac{5}{6}$ of a pizza, it does not matter which one you choose, because they are the same amount of pizza.

You can divide the top and bottom of a fraction by the same number, but you cannot subtract the same number from the top and bottom of a fraction!

$\frac{40}{48} = \frac{40-39}{48-39} = \frac{1}{9}$ ✗ Wrong!

Given the choice, a hungry person would be wise to choose $\frac{40}{48}$ of a pizza instead of $\frac{1}{9}$.

Dividing the top and bottom of a fraction by the same number leaves the fraction unchanged, and that is how you simplify fractions. **Subtracting** the same number from the top and bottom changes the value of the fraction, and is therefore an illegal simplification.

All this is review. But if you understand these basic fraction concepts, you are ahead of many Algebra II students! And if you can **apply these same concepts when variables are involved**, then you are ready to simplify rational expressions, because there are no new concepts involved.

As an example, consider the following:
Equation:

$$\frac{x^2 - 9}{x^2 + 6\text{x} + 9}$$

You might at first be tempted to cancel the common x^2 terms on the top and bottom. But this would be, mathematically, **subtracting** x^2 from both the top and the bottom; which, as we have seen, is an illegal fraction operation.

$\frac{x^2-9}{x^2+6\text{x}+9}$	$=\frac{\cancel{x^2}-9}{\cancel{x^2}+6x+9}$	$=\frac{-9}{6\text{x}+9}$	✗ **Wrong!**

To properly simplify this expression, begin by factoring both the top and the bottom, and **then** see if anything cancels.

Example: **Simplifying Rational Expressions**	
$\frac{x^2-9}{x^2+6\text{x}+9}$	**The problem**
$=\frac{(x+3)(x-3)}{(x+3)^2}$	**Always begin rational expression problems by factoring! This factors easily, thanks to** $(x+a)(x-a)=x^2-a^x$ and $(x+a)^2=x^2+2\text{ax}+a^2$
$=\frac{x-3}{x+3}$	**Cancel a common $(x+3)$ term on both the top and the bottom. This is legal because this term was multiplied on both top and bottom; so we are effectively dividing the top and bottom by $(x+3)$, which leaves the fraction unchanged**.

What we have created, of course, is an algebraic generalization:
Equation:

$$\frac{x^2 - 9}{x^2 + 6\text{x} + 9} = \frac{x - 3}{x + 3}$$

For any x value, the complicated expression on the left will give the same answer as the much simpler expression on the right. You may want to try one or two values, just to confirm that it works.

As you can see, the skills of **factoring** and **simplifying fractions** come together in this exercise. No new skills are required.

Rational Expression Concepts -- Multiplying Rational Expressions

This module covers the multiplication of rational expressions.

Multiplying fractions is easy: you just multiply the tops, and multiply the bottoms. For instance,

Equation:

$$\frac{6}{7} \times \frac{7}{11} = \frac{6 \times 7}{7 \times 11} = \frac{42}{77}$$

Now, you may notice that $\frac{42}{77}$ can be simplified, since 7 goes into the top and bottom. $\frac{42}{77} = \frac{42 \div 7}{77 \div 7} = \frac{6}{11}$. So $\frac{42}{77}$ is the correct answer, but $\frac{6}{11}$ is **also** the correct answer (since they are the same number), and it's a good bit simpler.

In fact, we could have jumped straight to the simplest answer first, and avoided dealing with all those big numbers, if we had noticed that we have a 7 in the numerator and a 7 in the denominator, and **cancelled** them before we even multiplied!

$$\frac{6}{\cancel{7}} x \frac{\cancel{7}}{11} = \frac{6}{11}$$

This is a great time-saver, and you're also a lot less likely to make mistakes.

When multiplying fractions...

If the same number appears on the top and the bottom, you can cancel it before you multiply. This works regardless of whether the numbers appear in the **same fraction or different fractions**.

But it's critical to remember that this rule **only applies when you are multiplying fractions**: not when you are adding, subtracting, or dividing.

As you might guess, all this review of basic fractions is useful because, once again, rational expressions work the same way.

Example:
Multiplying Rational Expressions

$\frac{3x^2-21x-24}{x^2-16} \cdot \frac{x^2-6x+8}{3x+3}$	**The problem**
$= \frac{3(x-8)(x+1)}{(x+4)(x-4)} \cdot \frac{(x-2)(x-4)}{3(x+1)}$	**Always begin rational expression problems by factoring! Note that for the first element you begin by factoring out the common 3, and then factoring the remaining expression.**
$= \frac{\cancel{3}(x-8)\cancel{(x+1)}}{(x+4)\cancel{(x-4)}} \cdot \frac{(x-2)\cancel{(x-4)}}{\cancel{3}\cancel{(x+1)}}$	**When multiplying fractions, you can cancel anything on top with anything on the bottom, even across different fractions**
$= \frac{(x-8)(x-2)}{x+4}$	**Now, just see what you're left with. Note that you could rewrite the top as $x^2 - 10x + 16$ but it's generally easier to work with in factored form.**

Dividing Rational Expressions

To divide fractions, you flip the bottom one, and then multiply.
Equation:

$$\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \cdot 3 = \frac{3}{2}$$

After the “flipping” stage, all the considerations are exactly the same as multiplying.

Example:
Dividing Rational Expressions

$\frac{\frac{x^2-3x}{2x^2-13x+6}}{\frac{x^3+4x}{x^2-12x+36}}$	This problem could also be written as: $\frac{x^2-3x}{2x^2-13x+6} \div \frac{x^3+4x}{x^2-12x+36}$. However, the symbol is rarely seen at this level of math. $12 \div 4$ is written as $\frac{12}{4}$.
$\frac{x^2-3x}{2x^2-13x+6} \times \frac{x^2-12x+36}{x^3+4x}$	Flip the bottom and multiply. From here, it’s a straight multiplication problem.
$= \frac{x(x-3)}{(2x-1)(x-6)} \times \frac{(x-6)^2}{x(x^2+4)}$	Always begin rational expression problems by factoring! Now, cancel a factor of x and an $(x-6)$ and you get...
$= \frac{(x-3)(x-6)}{(2x-1)(x^2+4)}$	That’s as simple as it gets, I’m afraid. But it’s better than what we started with!

Rational Expression Concepts -- Adding and Subtracting Rational Expressions
This module covers the addition and subtraction of rational expressions.

Adding and subtracting fractions is harder—but once again, it is a familiar process.

Equation:

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

The key is finding the **least common denominator**: the smallest multiple of both denominators. Then you rewrite the two fractions with this denominator. Finally, you add the fractions by **adding the numerators and leaving the denominator alone**.

But how do you find the least common denominator? Consider this problem:

Equation:

$$\frac{5}{12} + \frac{7}{30}$$

You could probably find the least common denominator if you played around with the numbers long enough. But what I want to show you is a **systematic method** for finding least common denominators—a method that works with rational expressions just as well as it does with numbers. We start, as usual, by factoring. For each of the denominators, we find all the **prime factors**, the prime numbers that multiply to give that number.

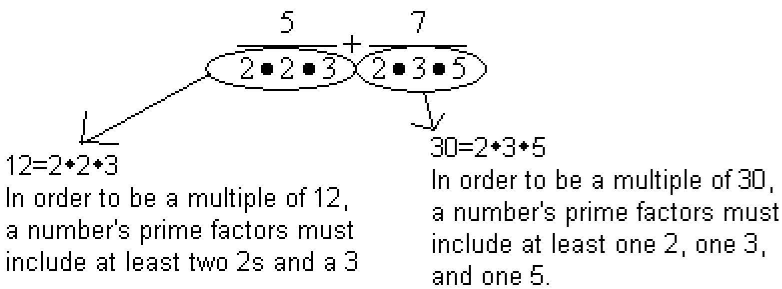
Equation:

$$\frac{5}{2 \cdot 2 \cdot 3} + \frac{7}{2 \cdot 3 \cdot 5}$$

If you are not familiar with the concept of prime factors, it may take a few minutes to get used to. $2 \times 2 \times 3$ is 12, broken into its **prime factors**: that

is, it is the list of prime numbers that multiply to give 12. Similarly, the prime factors of 30 are $2 \times 3 \times 5$.

Why does that help? Because $12 = 2 \times 2 \times 3$, any number whose prime factors include two 2s and one 3 will be a multiple of 12. Similarly, any number whose prime factors include a 2, a 3, and a 5 will be a multiple of 30.



The least common denominator is the smallest number that meets both these criteria: it must have two 2s, one 3, and one 5. Hence, the least common denominator must be $2 \times 2 \times 3 \times 5$, and we can finish the problem like this.

Equation:

$$\frac{5}{2 \cdot 2 \cdot 3} + \frac{7}{2 \cdot 3 \cdot 5} = \frac{5\underline{5}}{(2 \cdot 2 \cdot 3)\underline{5}} + \frac{7\underline{2}}{(2 \cdot 3 \cdot 5)\underline{2}} = \frac{25}{60} + \frac{14}{60} = \frac{39}{60}$$

This may look like a very strange way of solving problems that you've known how to solve since the third grade. However, I would urge you to spend a few minutes carefully following that solution, focusing on the question: **why is $2 \times 2 \times 3 \times 5$ guaranteed to be the least common denominator**? Because once you understand that, you have the key concept required to add and subtract rational expressions.

Example:
Subtracting Rational Expressions

$\frac{3}{x^2+12x+36} - \frac{4x}{x^3+4x^2-12x}$	**The problem**
$= \frac{3}{(x+6)^2} - \frac{4x}{x(x+6)(x-2)}$	**Always begin rational expression problems by factoring! The least common denominator must have two $(x+6)$ s, one x , and one $(x-2)$.**
$= \frac{3(x)(x-2)}{(x+6)^2(x)(x-2)} - \frac{4x(x+6)}{x(x+6)^2(x-2)}$	**Rewrite both fractions with the common denominator.**
$= \frac{3(x)(x-2)-4x(x+6)}{x(x-2)(x+6)^2}$	**Subtracting fractions is easy when you have a common denominator! It's best to leave the bottom alone, since it is factored. The top, however, consists of two separate factored pieces, and will be simpler if we multiply them out so we can combine them.**
$= \frac{3x^2-6x-(4x^2+24x)}{x(x-2)(x+6)^2}$	**A common student mistake here is forgetting the parentheses. The entire second term is subtracted; without the parentheses, the $24x$ ends up being added.**
$= \frac{-x^2-30x}{x(x-2)(x+6)^2}$	**Almost done! But finally, we note that we can factor the top again. If we factor out an x it will cancel with the x in the denominator.**

$= \frac{-x-30}{(x-2)(x+6)^2}$	**A lot simpler than where we started, isn't it?**

The problem is long, and the math is complicated. So after following all the steps, it's worth stepping back to realize that even this problem results simply from the two rules we started with.

First, always factor rational expressions before doing anything else.

Second, follow the regular processes for fractions: in this case, the procedure for subtracting fractions, which involves finding a common denominator. After that, you subtract the numerators while leaving the denominator alone, and then simplify.

Rational Expression Concepts -- Rational Equations
This module introduces rational expressions in equations.

Rational Equations

A **rational equation** means that you are setting two rational **expressions** equal to each other. The goal is to solve for x; that is, find the x value(s) that make the equation true.

Suppose I told you that:
Equation:

$$\frac{x}{8} = \frac{3}{8}$$

If you think about it, the x in this equation has to be a 3. That is to say, if x=3 then this equation is true; for any other x value, this equation is false.

This leads us to a very general rule.

A very general rule about rational equations

If you have a rational equation where the **denominators** are the same, then the **numerators** must be the same.

This in turn suggests a strategy: find a common denominator, and then set the numerators equal.

Example: Rational Equation

Example: Rational Equation	
$\frac{3}{x^2+12x+36} = \frac{4\text{x}}{x^3+4\text{x}^2-12x}$	Same problem we worked before, but now we are equating these two fractions, instead of subtracting them.
$\frac{3(x)(x-2)}{(x+6)^2(x)(x-2)} = \frac{4\text{x}(x+6)}{x(x+6)^2(x-2)}$	Rewrite both fractions with the common denominator.
$3x(x-2) = 4x(x+6)$	Based on the rule above—since the denominators are equal, we can now assume the numerators are equal.
$3x^2 - 6$ x$= 4x^2 + 24x$	Multiply it out
$x^2 + 30x = 0$	What we're dealing with, in this case, is a quadratic equation. As always, move everything to one side...
$x(x+30) = 0$	...and then factor. A common mistake in this kind of problem is to divide both sides by x; this loses one of the two solutions.
x$=0$ **or** x$=-30$	Two solutions to the quadratic equation. However, in this case, $x = 0$ is not valid, since it was not in the domain of the original right-hand fraction. (Why?) So this problem actually has only one solution, $x = -30$.

As always, it is vital to remember what we have found here. We started with the equation $\frac{3}{x^2+12x+36} = \frac{4\mathrm{x}}{x^3+4\mathrm{x}^2-12x}$. We have concluded now that if you plug $x = -30$ into that equation, you will get a true equation (you can verify this on your calculator). For any other value, this equation will evaluate false.

To put it another way: if you graphed the functions $\frac{3}{x^2+12x+36}$ and $\frac{4\mathrm{x}}{x^3+4\mathrm{x}^2-12x}$, the two graphs would intersect at one point only: the point when $x = -30$.

Rational Expression Concepts -- Dividing Polynomials

Simplifying, multiplying, dividing, adding, and subtracting rational expressions are all based on the basic skills of working with fractions. Dividing polynomials is based on an even earlier skill, one that pretty much everyone remembers with horror: long division.

To refresh your memory, try dividing $\frac{745}{3}$ by hand. You should end up with something that looks something like this:

```
   248 r1
3|745
  6
  14
  12
   25
   24
    1
```

So we conclude that $\frac{745}{3}$ is 248 with a remainder of 1; or, to put it another way, $\frac{745}{3} = 248\frac{1}{3}$.

You may have decided years ago that you could forget this skill, since calculators will do it for you. But now it comes roaring back, because here is a problem that your calculator will not solve for you: $\frac{6x^3-8x^2+4x-2}{2x-4}$. You can solve this problem in much the same way as the previous problem.

Example:

$\frac{6x^3-8x^2+4x-2}{2x-4}$	**The problem**

$2x-4\overline{)6x^3-8x^2+4x-2}$	**The problem, written in standard long division form.**
$\begin{array}{r} 3x^2 \\ 2x-4\overline{)6x^3-8x^2+4x-2} \end{array}$	**Why $3x^2$? This comes from the question: "How many times does $2x$ go into $6x^3$?" Or, to put the same question another way: "What would I multiply $2x$ by, in order to get $6x^3$?" This is comparable to the first step in our long division problem: "What do I multiply 3 by, to get 7?"**
$\begin{array}{r} 3x^2 \\ 2x-4\overline{)6x^3-8x^2+4x-2} \\ \underline{6x^3-12x^2} \\ 4x^2 \end{array}$	**Now, multiply the $3x^2$ times the $(2x-4)$ and you get $6x^3-12x^2$. Then subtract this from the line above it. The $6x^3$ terms cancel—that shows we picked the right term above! Note that you have to be careful with signs here.** $-8x^2- \ -12x^2$ **gives us positive $4x^2$.**
$\begin{array}{r} 3x^2 \\ 2x-4\overline{)6x^3-8x^2+4x-2} \\ \underline{6x^3-12x^2} \\ 4x^2+4x \end{array}$	**Bring down the $4x$. We have now gone through all four steps of long division—divide, multiply, subtract, and bring down. At this point, the process begins again, with the question "How many times does $2x$ go into $4x^2$?"**
$\begin{array}{r} 3x^2+2x+6 \ r\ 22 \\ 2x-4\overline{)6x^3-8x^2+4x-2} \\ \underline{6x^3-12x^2} \\ 4x^2+4x \\ \underline{4x^2-8x} \\ 12x-2 \\ \underline{12x-24} \\ 22 \end{array}$	**This is not the next step...this is what the process looks like after you've finished all the steps. You should try going through it yourself to make sure it ends up like this.**

Polynomial Division

So we conclude that $\frac{6x^3-8x^2+4x-2}{2x-4}$ is $3x^2+2x+6$ with a remainder of 22, or, to put it another way, $3x^2+2x+6+\frac{22}{2x-4}$.

Checking your answers

As always, checking your answers is not just a matter of catching careless errors: it is a way of making sure that you **know what you have come up with**. There are two different ways to check the answer to a division problem, and both provide valuable insight

The first is by plugging in numbers. We have created an algebraic generalization:

Equation:

$$\frac{6x^3 - 8x^2 + 4x - 2}{2x - 4} = 3x^2 + 2x + 6 + \frac{22}{2x - 4}$$

In order to be valid, this generalization must hold for $x = 3$, $x = -4$, $x = 0$, $x = \varpi$,or any other value except $x = 2$ (which is outside the domain). Let's try $x = 3$.

Checking the answer by plugging in $x = 3$

Equation:

$$\frac{6(3)^3 - 8(3)^2 + 4(3) - 2}{2(3) - 4} = 3(3)^2 + 2(3) + 6 + \frac{22}{2(3) - 4}$$

Equation:

$$\frac{162 - 72 + 12 - 2}{6 - 4} = 27 + 6 + 6 + \frac{22}{6 - 4}$$

Equation:

$$\frac{100}{2} = 39 + \frac{22}{2}$$

Equation:

$$50 = 39 + 11$$

The second method is by multiplying back. Remember what division is: it is the opposite of multiplication! If $\frac{745}{3}$ is 248 with a remainder of 1, that means that $248 \cdot 3$ will be 745, with 1 left over. Similarly, if our long division was correct, then $3\mathrm{x}^2 + 2\mathrm{x} + 6\ \ (2\mathrm{x} - 4) + 22$ should be $6\mathrm{x}^3 - 8\mathrm{x}^2 + 4\mathrm{x} - 2$.

Checking the answer by multiplying back

Equation:

$$3\mathrm{x}^2 + 2\mathrm{x} + 6\ \ (2\mathrm{x} - 4) + 22$$

Equation:

$$= \ \ 6\mathrm{x}^3 - 12x^2 + 4\mathrm{x}^2 - 8\mathrm{x} - 24 \ \ + 22$$

Equation:

$$= 6\mathrm{x}^3 - 8\mathrm{x}^2 + 4\mathrm{x} - 2$$